Brown, Lewis and Epstein's 1937 Antenna Ground System Paper - Theoretical Aspects Comments Bordering on a Critique - N6MW December 2015

Introduction

My prior write up [N6MW, 2015] on Brown et al.'s paper (hereafter referred to as Brown [Brown, 1937]) concerned the interpretation of the experimental aspects of the work. However, some of the experiments are cast in terms of the theoretical elements which are discussed at the beginning of the article in Section II. Some aspects of this section prompt observations that follow here.

Loop Antenna Current and Radiation Resistance

The initial part of the Brown paper discussions are for a vertical radiator over a perfectly conducting infinite flat earth. Figure 1 is for loop radiation resistance, Rr(loop), of a vertical antenna of height G = $2\pi a/\lambda$ in radians. The radiation resistance, that we know and love, at the base is then related by Eq (1) as:

$$R_r(\text{base}) = R_r(\text{loop})/\sin^2 G.$$

The meaning of "loop" seems to be of obscure origins, but it refers to the location(s)/current on the antenna where the current is a maximum (a node), assuming a sinusoidal current distribution. In the case of an antenna shorter than G of $\pi/2$ (or $\lambda/4$), the loop current is what it would have been if the antenna were extended to $\pi/2$. Note that for a quarter wave vertical (or $(2m+1)\lambda/4$ for any integer m) the base and loop current are the same.

The apparent beauty of the loop current concept is that the loop radiation resistance, which is "referred to" the loop current, is calculable in terms of (transcendental) functions [see Jasik,for example] that are tabulated . This Rr(loop) is plotted in Fig 1. A charm of Rr(loop) may be that it never becomes infinite, in contrast to Rr(base). However, Rr(loop) does not, in general, correspond to any physical measurement.

The plot in Fig 1 should be used with some caution for the small G end of the curve. Brown says correctly that Rr(base) ~ $10G^2$ at smaller G so, for example, at 30 degrees (or 0.53 rad) Rr(base) ~2.8 ohms making Rr(loop) ~ 0.8 ohms from Eq (1) above. However the plot suggests ~ 10 ohms. At 90 degrees, the plot gives Rr(loop) of ~ 47 which is well off the correct 36.5.

I conclude that the information in Fig 1 pertaining to a loop radiation resistance is poorly plotted plus of no particular value here. The good news is that figure is never actually used in the paper. The plot there of F appears to be the result from Eq (4) when the Rr(base) from Fig 2 is used with Power=1000 W. So we will ignore the loop resistance in Fig 1 and use Fig 2 as the source of the radiation resistance of a vertical over a perfectly conducting earth. At G of 90 degrees, or $\pi/2$, the base radiation resistance Rr is 36.5 ohms, ~35.6 at 88 degrees and ~1.5 ohms at 22 deg. These seem to be conventional wisdom.

Vertical over Perfect Conductor

The azimuthly integrated radial current peak amplitude, all on the surface, at a given range for a vertical over an infinite conducting plane is given by Eq (5) below. Note that (only) for the case of G = 90 degrees (quarter wave antenna) this integrated current is independent of range – and it extends to infinity. The peak surface current density then falls off like 1/range while the integrated peak current over azimuth is constant with range. As shown in Fig 4, this behavior is true at larger ranges for all G.

Of course, for a perfect conductor, there are no losses so all the power is ultimately radiated.

$$|I_z| = \frac{I_0}{\sin G}\sqrt{1 + \cos^2 G - 2\cos G \cos k(r_2 - x)} \qquad (5)$$

I_0 = current at the base of the antenna

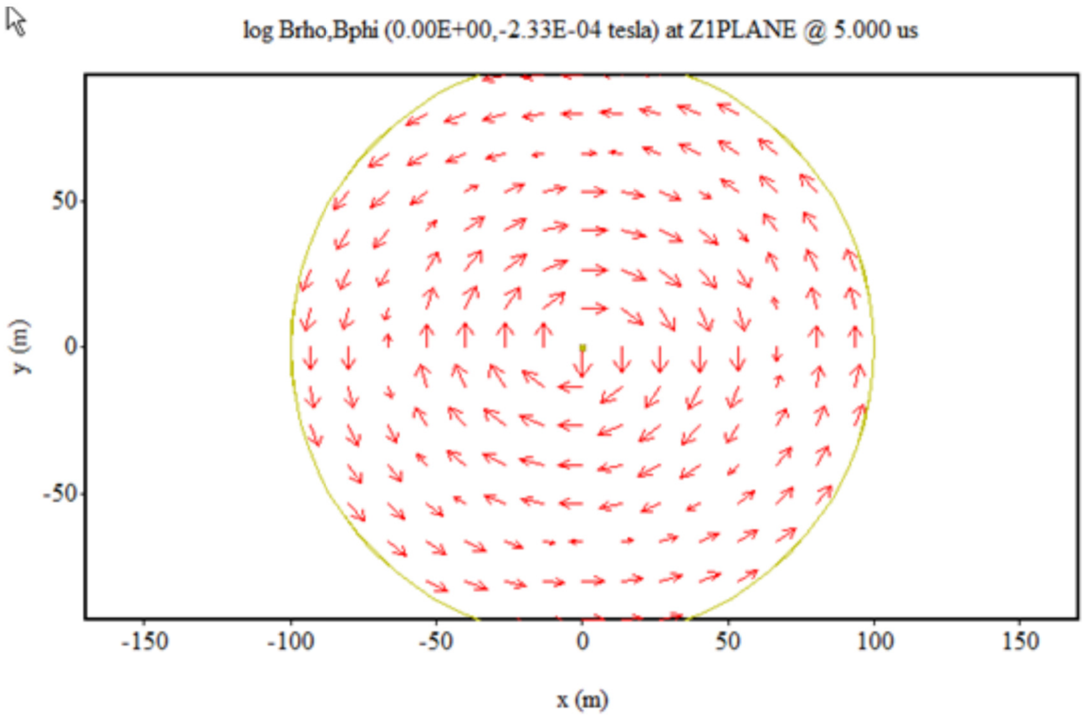
$r_2 = \sqrt{a^2 + x^2}$

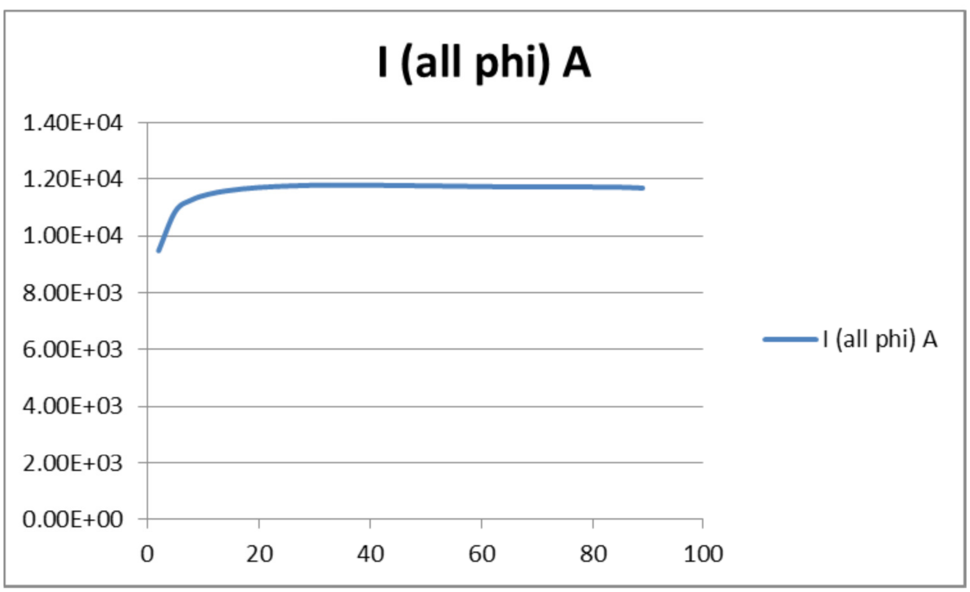
$k = 2\pi/\lambda$.

This might leave you with the impression, particularly for G=π/2, that the current flows continuously in the same direction over all radii – **this is not the case**. The result in Eq (5) is for the peak current over time at that range BUT the peak currents do no occur at the same times at different ranges and there are current zones of alternating signs, apparently to infinity, with zone size dictated basically by the wavelength.

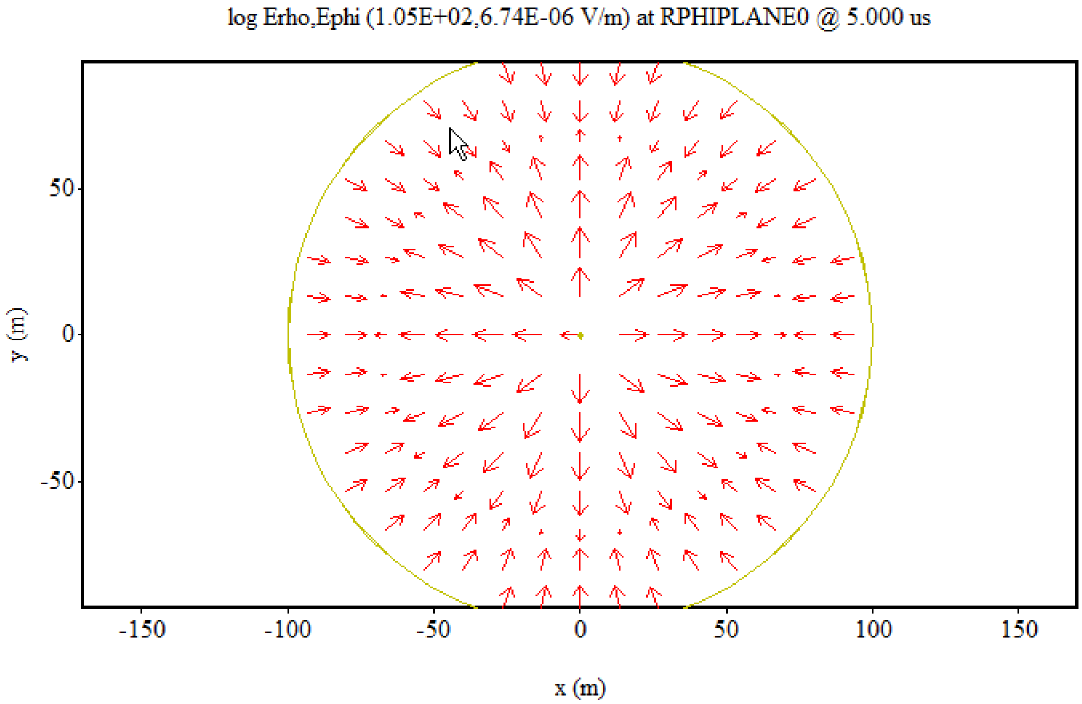
As an illustration, we will provide the results from a Maxwell equation solver for a G~π/2 vertical over a good conducting earth in cylindrical coordinates as shown in the next figures. First there is a 3D construction, in cylindrical coordinates, to provide the scale of the vertical (white) and surface (ignore funny shading).

Next there is a snapshot of the (nearly azimuthal) magnetic B field just above the surface where the vector lengths are scaled logarithmically – so the lengths of smaller amplitudes are exaggerated. It turns out that this field largely defines the fields and currents at points under the surface [see J. D. Jackson, 1975] for large but finite conductivity. The current (perpendicular to B and radial), is nearly confined to the surface layer due to the small skin depth, and the magnitude can be calculated in a general way by the very sophisticated and expensive full-up commercial finite difference Maxwell (and Lorentz if needed) equation solver called MAGIC [MAGIC]. Note that the direction (along the azimuth) and amplitude of B oscillates as you go along a radial. The conduction current peak magnitude with range is shown in the second plot. This current is directly proportional to the B field just above the surface (thanks Prof. Ampere) but you must look over a cycle to find the peak. The arbitrary normalization here is for an antenna input current of 12kA but the I(r) pictured below is very much like Brown's Fig 4 for G of 88 deg as expected, except at small radii where the grid size restriction of MAGIC probably limits the accuracy. The peak amplitude of integrated currents over all azimuths flowing near the ground surface is nearly independent of radius.

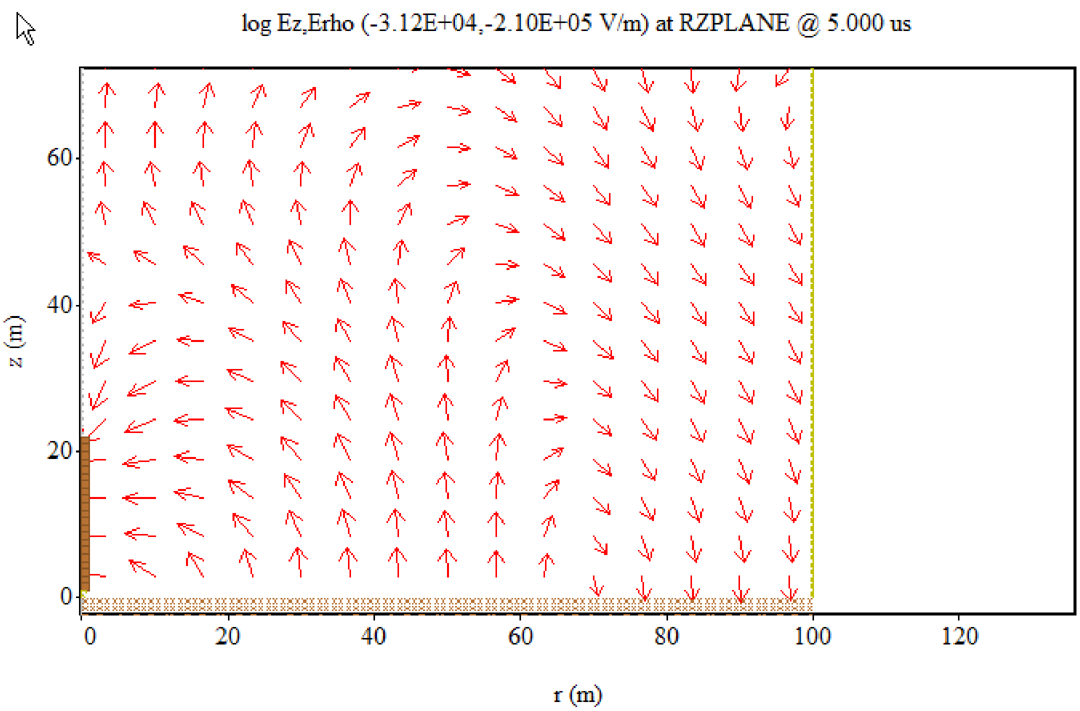




As a further illustration, next we provide a snapshot in time of the vector horizontal E field just below the surface. This is for the case of a not quite perfectly conducting earth surface (sigma =1 out to 100 m) and a vertical G~π/2 antenna (~22 m in simulation) . (If perfectly conducting, E is zero below the surface so not a useful diagnostic.) We see that the E is radial and, like B, oscillating. The current can then be obtained from E as σE (thanks Prof. Ohm). There is no significant displacement current in the ground due to the large sigma.



Finally we show the E vector in a horizontal plane through the antenna. The expected umbrella-ish shape with E perpendicular to the plane and the vertical antenna wire are evident. Again the vector lengths are on a log scale so those vectors away from the conductors are smaller than they appear.



The information above is probably not at all controversial due to the use of a very good conductor - but it serves to provide some verification of the theory and the software tool plus provides some understanding of the physics of this geometry. Now we move on to the good stuff.

Not so Perfect Conductivity and Radials

Beginning with the paragraph containing Eq (8), Brown transitions to considering a realistic imperfectly conducting earth which can have perfectly conducting radial wires of infinite length of different number (n) and wire radius (r) near the surface. Eq (8) postulates that the ratio of the total currents in the imperfect earth and the total current in radial wires at a given range (x) is

$$I_e/I_w = j\gamma_e \cdot 4\pi^2 \cdot 10^{-9} f c^2 \left\{ \log \frac{c}{r} - 0.5 \right\} \quad (8)$$

where

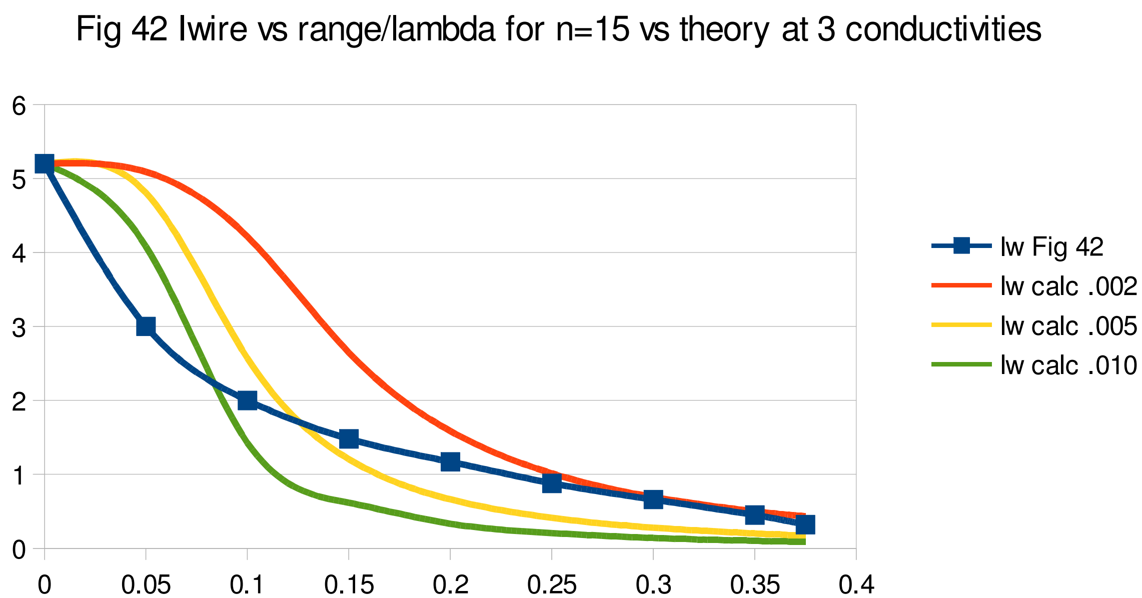
γ_e = earth conductivity (mhos per centimeter cube)
f = frequency (cycles per second)
x = distance from antenna (centimeters)
n = number of equally spaced radial wires
$c = \pi x/n$
r = radius of the wire in the ground system.

In the paper here is no reference to the origins of this relation, yet it is the driver for all subsequent theoretical discussion. The equation might be from Brown's dissertation but the authors do no say that and they offer no justification. Aside from having the right units and some plausible sounding parametric dependences, it is difficult (for me) to concoct an argument for its correctness. At least some of it likely comes from empirical fits to approximate calculations, it being from pre-computer 1937 back when actual thinking was required, and you know how painful that can be. The part of the dependence on range in the curly brackets causes the expression to go to zero at a small but finite range suggesting zero Ie inside that range, likely because the wire radials cover much of the ground – this part is clearly empirical. Generally the {} expression is a pretty slow function of range and wire radius. The range where the {} expression vanishes is at the point where the area covered by the wires is about 60% of the area at that radius. We assume that the equation should not be used inside a radius where {} becomes negative. Note that the "log" in the equation refers to the natural log to the base e (often written ln) and not the log to the base 10, as becomes evident when comparing the equation with subsequent plots.

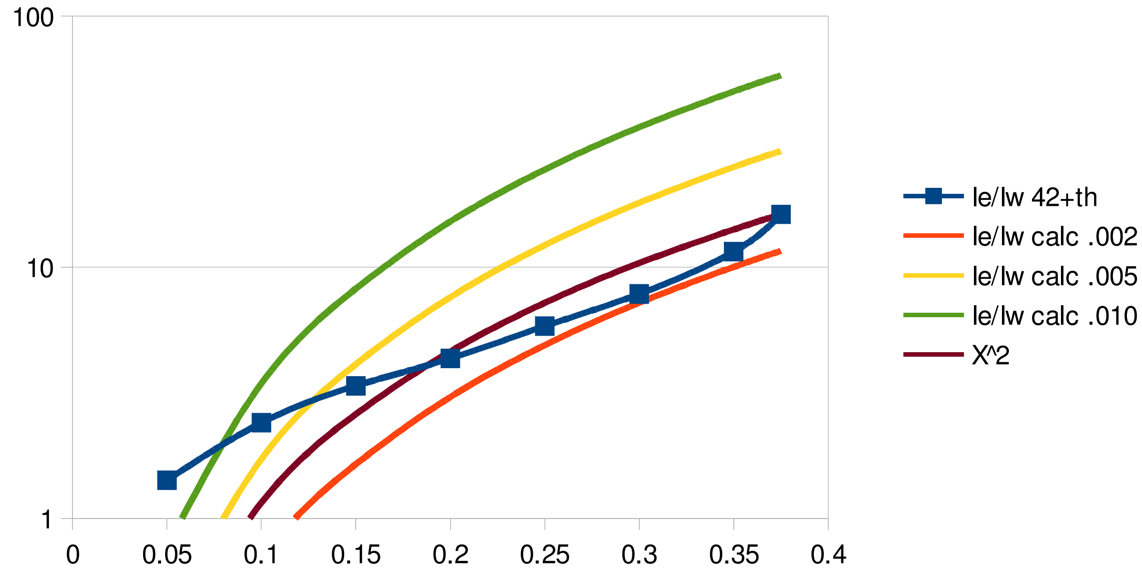
The coefficient in front of the {} goes like the conductivity (sensible) and $(\text{range}/n)^2$ which is not intuitive (again to me). Note that the ratio of conduction current to displacement current densities in the ground is $j\sigma/(2\pi f\varepsilon)$ where ε is the absolute dielectric constant (8.85e-12 in MKS units for a vacuum). This expression can be rewritten as $j2\pi\mu f\sigma(\lambda/2\pi)^2$ where the permeability μ is $4\pi*10^{-9}$ in Brown's units. This expression resembles Brown's Eq (8), aside from the $\lambda/2\pi$ vs x, but that may not be too surprising since there are a limited number of ways making the units right. The somewhat faster that x^2 dependence of Ie/Iw is a bit of a curiosity since the ratio of the area of ground versus that wire covered goes like just x at any radius. So (if (8) is correct) this means that the wires are rapidly increasing effective at collecting the ground current at small radii - this is well beyond the obvious.

Brown Experimental Wire Current Example

Before exploring the implication of MAGIC calculations, we note that Brown includes one experimental plot (Fig 42) that shows actual measurements of the wire current with range for 135 ft radials at 3 MHz. Brown makes no attempt to compare this with the theoretical plots provided before. If we assume that the dependence of Ie/Iw truly has the factor j, a measurement of Iw from the experimental data can be directly compared with the Eq (8) theory for Iw. It turns out that most of the Iw plots of Fig 42 contain some funny wiggles but the n=15 case is more pleasing. [Earlier in the paper Brown in loosely describing the current effects, perhaps in anticipation of Fig 42, says "As the current flows in toward the antenna, it is continually added to by more displacement currents (i.e., *from the E field above the ground, N6MW*) flowing into the earth. It is not necessarily true that the earth currents will increase because of this additional displacement current, since all the various components differ in phase." One might question this convenient statement in relation to the data and theory.] So let us take just that n=15 case and compare with Eq (8) for a likely set of conductivities (.002, .005, and .010 S/m), two of which Brown uses for examples (but in his cm units). Wire current vs range/λ from the experimental data is shown below compared with that obtained from Eq (8) for three conductivities, all at n=15.



Clearly the shape of the Iw data curve is substantially different from theory although the range over which the amplitude becomes small is similar for both. Another way to judge is to look at Ie/Iw "data" (Ie/Iw is not measured, only Iw, so some theory must be invoked to get the ratio) versus theory to compare the slightly faster than X^2 clearly against the range/lambda axis. This is shown below on a log scale with an X^2 curve added to help the eye evaluate. Again the "data" curve is clearly different from the theory curves, which tend to run nearly "parallel" with X^2. However, you might stretch a bit and argue that the data curve would not be hugely different from a conductivity of ~ 0.003 S/m. Still the difference in shape is not comforting. In Brown's defense, it should be pointed out that, strictly speaking, the theory is said to apply only for infinitely long radials – however, the current at the end of the experimental radials is pretty small. Ratio of earth to wire current vs range/λ is shown below.



It should be noted that the radials are of finite length may well impact this result – plus the theory takes the total current to be the same as that for an infinitely extended conducting ground plane as indicated before.

Displacement Current and Skin Depth Assumptions

As part of the Brown discussion, it is implied that the displacement current in the earth (εd**E**/dt) is insignificant compared to the conduction current (σ**E**) in the earth and so the quantity Ie is the earth conduction current, now said to be flowing radially and integrated down deep enough below the surface to capture all the current there. As regards the units employed by Brown, the conductivity γe appears to be the same as the more conventional mhos/cm and the wire radius must be in cm (I have taken it to have a 0.17 cm radius based on experimental value from the text.)

Again there is the further postulation that, for fixed driving power, the vector sum of the earth current (Ie) and wire current (Iw) is exactly the same as the total current for the perfectly conducting ground case (Ix) at a given radius, even though the ground current is no longer just on the surface. Further note that Ie and Iw are 90 degrees offset so they are postulated to be out of phase as denoted by the "j." For the special case G=90deg, the peak amplitude of current Ix is independent of range. The total current at the base is denoted by Io and taken to be the same as that for a perfectly conducting earth.

See the paragraph before Eq (5) for a qualitative description of the how the currents are set up for the case of an infinitely large pretty good conducting surface (not the wire radial case). A short version of that story is the **E** fields just above the ground surface, acting by its associated displacement current ε_od**E**/dt, "flows" vertically into the ground and this is added to the ground conduction current that flows radially through the ground back toward the antenna base. Note that the displacement current "flow" into the conductor really corresponds to a changing charge density on the surface which comes from a stopped flow of wire conduction current at the wire edge. Part of this total current flows in the radial wires although no description of provided as to just how some of the flow goes to the wires. But if there were more radial wires, the division of the current would, we assume, favor the wires at a given range. Although not stated this way by the authors, Eq (8) provides the current ratio at a given range and the Ie portion is reduced at smaller ranges while Iw increases there suggesting that the earth current flows more into the wires as the wires cover more and more of the surface nearer the base. However, the authors never make any statement about the physics of the flow into the wires beyond the implication of the proclaimed Eq (8). Of course, the description here does not contain enough information to allow actual calculation of the currents since an internally consistent Maxwell equation solution satisfying all the boundary conditions must first be in place.

At this point in the paper a series of examples, based on Eq (8), of theoretical current distributions with range in terms of range/lambda are provided for 3 and 1 MHz and γe of 0.2e-4 and 1.0e-4 mhos/cm, Brown calls this mhos/cm^3 which correspond to 0.002 and 0.01 mhos/m (or S/m), plus for G of 88 to 22 degrees and for radial number n of 15, 30 , 60 and 113. Note again that Eq (8) assumes the wires are infinitely long although this may not be very important for cases where little current is at the ends of the wires from theory. The 88 degree case is instructive since it is very close to a common case of interest, a vertical height of lambda/4. The authors never suggest that the dielectric constant of the earth plays any role and in fact it is not even mentioned in the article. For realistic soil, where the dielectric constant can be over 10, some of Brown's lower conductivity examples would contradict the assumption of insignificant displacement current in the ground. In estimating the power loss to the finite conductivity of the earth, the article assumes that the total ground current can be approximated as being radial and uniform down to a skin depth but zero below. This is probably not a bad assumption although the details of current behavior with depth are rather more complex. Jackson shows how the phase of the current, as well as amplitude, vary with depth.

First let us examine the skin depth (s below) and ratio of the ground conduction to ground displacement currents. The conduction to displacement current ratio is σ/(ω*ε) and the two are out of phase by 90 degrees. The skin depth used by Brown is the high conductivity value (from assuming conduction current >> displacement current) of

$$s = \frac{1}{\sqrt{\pi \mu \gamma_e f}} \text{ (cm)}$$

$\mu = 4\pi \cdot 10^{-9}$
γ_e = conductivity of earth (mhos per cm^3)
f = frequency (cycles per second).

The expression for not-so-high conductivity is well known but more complicated and it depends on the dielectric constant. The following table shows the ratios of conduction current to displacement current, Ic/Id, and Brown's approximate skin depth versus the true value both for a relative dielectric constant of 1 (implicitly assumed by Brown) and for a large but realistic value of 13. Often here the approximation of insignificant displacement currents in the ground and the high conductivity skin depth are okay although they become questionable for the lower conductivity of .002 and higher frequency of 3 MHz examples when a realistic dielectric constant is used.

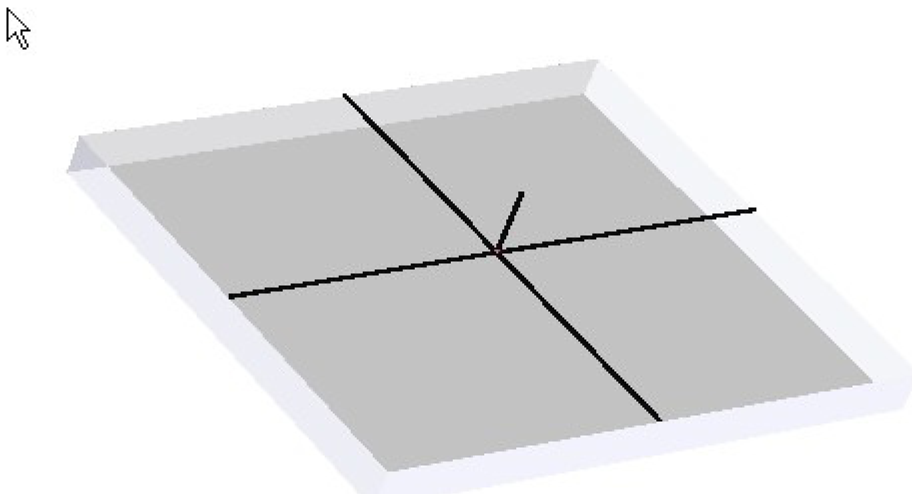
sigma (S/m)	eps	freq (Hz)	s (m)	Ic/Id
0.01	1	3000000	2.9	59.9
0.002	1	3000000	6.8	12.0
0.01	13	3000000	3.2	4.6
0.002	13	3000000	10.4	0.9
0.01	1	1000000	5.0	179.8
0.002	1	1000000	11.4	36.0
0.01	13	1000000	5.2	13.8
0.002	13	1000000	13.4	2.8

Some spot checks of the plotted values of Ie and Iw versus Eq (8) were made for several 22 and 88 degree cases. This requires using the expression for Ix (G,x) in Eq (5) together with Rr(G) in Fig 2. The plots are in largely in agreement with the equations, albeit with some question of the accuracy of the hand made plots.

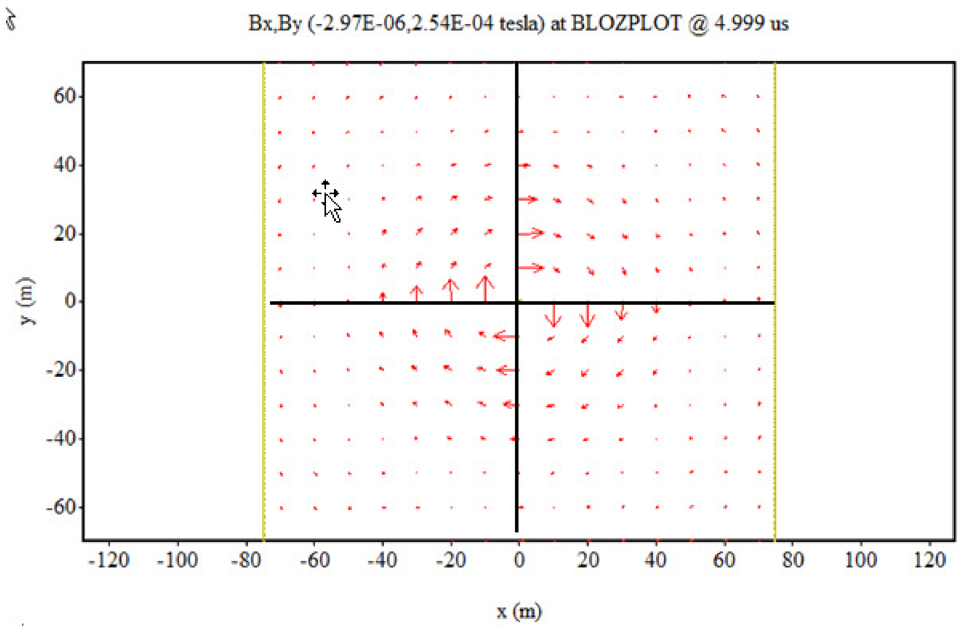
In addition, using Brown's expression for skin depth plus the theoretical ground current, an evaluation of the power loss shown in Fig 18 using the assumptions in the paper for the n=15, 88 degree case was done and good agreement was found there as well.

The Paper Results, Primarily Eq (8), Compared With A Direct Solution To Maxwell's Equation
In view of the lack of information about the basis for Eq (8), I have attempted to make a comparison with results from a full-up commercial Maxwell equation solver, MAGIC. For the most part, the accuracy of the results is limited only by the gridding resolution, and so number of cells (which is limited to one million in my case, and spatial extent used, together with the trade off of how long you are prepared to wait for the answer. This is certainly not a trivial limitation and the available gridding is more coarse than is desirable. For example, the radial wires are taken to be 1 meter square in a grid with 1 meter cells in the interesting region. Still the results look sensible.

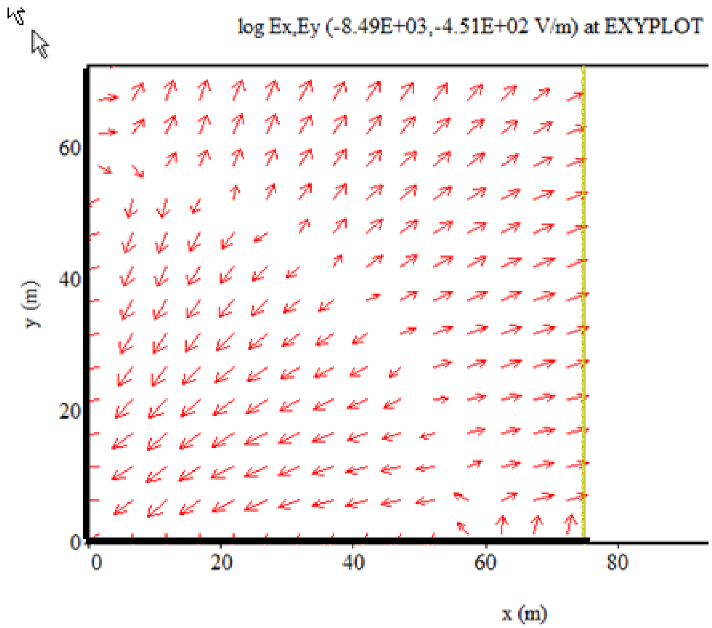
Radials Included in the Simulation
Earlier discussion in this brief report showed MAGIC calculations for a vertical over a large perfectly conducting earth. For MAGIC calculations with radials, the best available case is for cartesian geometry with four “infinite” radials of 75 m extending to edge of the simulation ground surface. The top of the radials is at the ground surface so the radials are embedded in the ground on three sides. A frequency of 3 MHz is used with a ground conductivity of 0.0026 S/m, a ground depth of 15 m (so bit more than 2 skin depths) and a relative dielectric constant of 1. The ground area is 150X150 m and the outer boundaries are all transmissive (approximately, as possible). This case satisfies Brown's assumption about small displacement current in the ground and the objective here is to compare as directly as possible with Brown's corresponding results in Eq (8) for his G=88 deg case,
limited to a small number of radials. The ratio of ground conduction to displacement currents is about 12 with a skin depth of about 6 meters so displacement current in the ground is small. The geometry is shown below. Unfortunately it is necessary to use very large (1mX1m cross section) radial wires and vertical antenna element to limit the number of cells. The good news is that the theory says the dependence on the wire radius is very mild.



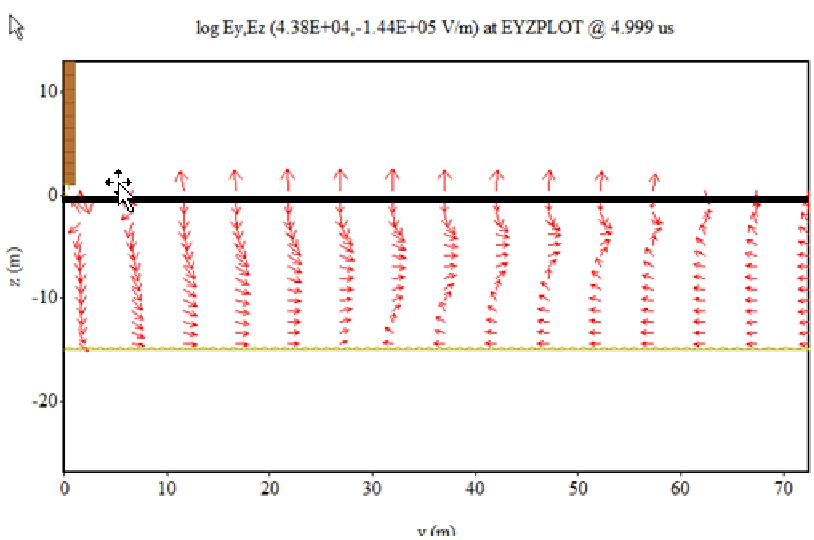
With radials, the B field near the surface remains largely azimuthal but the field above the radials is now much larger than that over the ground. Note this snapshot plot uses a linear scale for vector lengths so the range of field amplitudes to make clear the large fields above the radials (black lines).



Without radials, the **E** fields at the surface are purely radial as seen before. When radials are added, the **E** fields near the radial and antenna wires are forced to be perpendicular to the conductors. This causes the **E** fields away from the radials to curve toward those radials. In the snapshot below for a plane just below the surface, a log scale for vector lengths is used so the range of lengths is greater than it appears.



For a final snapshot plot, the **E** fields in the ground under the radials show a more complex behavior than might be suggested by just considering the skin depth. In the ground **E** is proportional to the conduction current density, **J**, in this case so you can see that the flow up to the radial depends on the range from the antenna in a complicated way. Again a log scale for vector lengths is used.

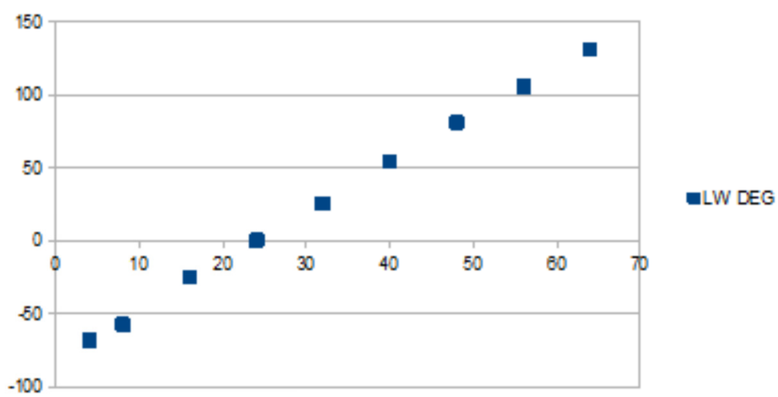


Wire and Ground Currents in the Simulation

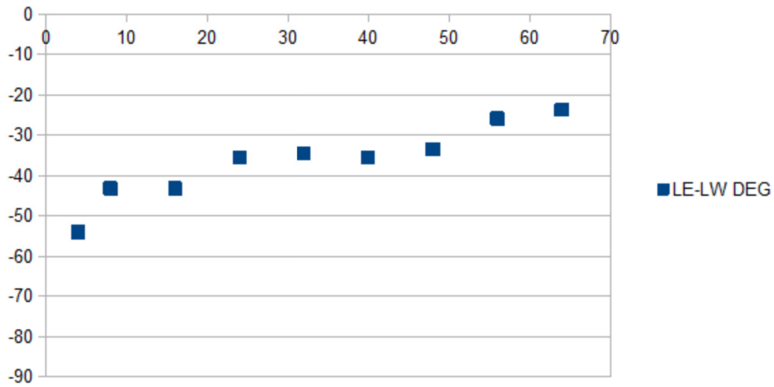
To measure the wire currents in MAGIC, it has a diagnostic capability of using Ampere's law which says the integral of H around a loop (**H.dl**) equals the current through the loop. Generally this would be the sum of the conduction and displacement current but the displacement current is not significant here. Note that **H** = **B**/μ. This is then used at a set of radii with a square loop around the radial wire to find the magnitude and phase of the wire current as a function of range.

The earth current can also be found using (**H.dl**) rectangular loops (with some care) that extend from above the ground surface more than two skin depths down to the bottom of the mildly conducting ground. This was done along two azimuths at 45 and 22.5 degrees between the wire radials (which are at 0, 90, 180 and 270 deg). Due to the use of cartesian coordinates, coordinated pairs of loops in two directions are needed to estimate radial current - and the measurement locations are only at approximately the same radii as used for the wire measurements due to the gridding. It turns out that the 22.5 degree values are pretty similar to those from 45 degrees (for which the current is radial by symmetry) until you get near the center. This is not surprising since the ground current there are expected to start to flow into the radials. See the two plots above. The radial current along the 45 degree azimuth is first transformed to the current density per meter along a nominal 1 meter arc across that azimuth for each range. The total earth current is then found by multiplication with the circumference (effectively integrating along that circumference). However, using the full circumference is somewhat of an overestimate since the radial wires take up a small part of it. So the earth currents may be a little too big.

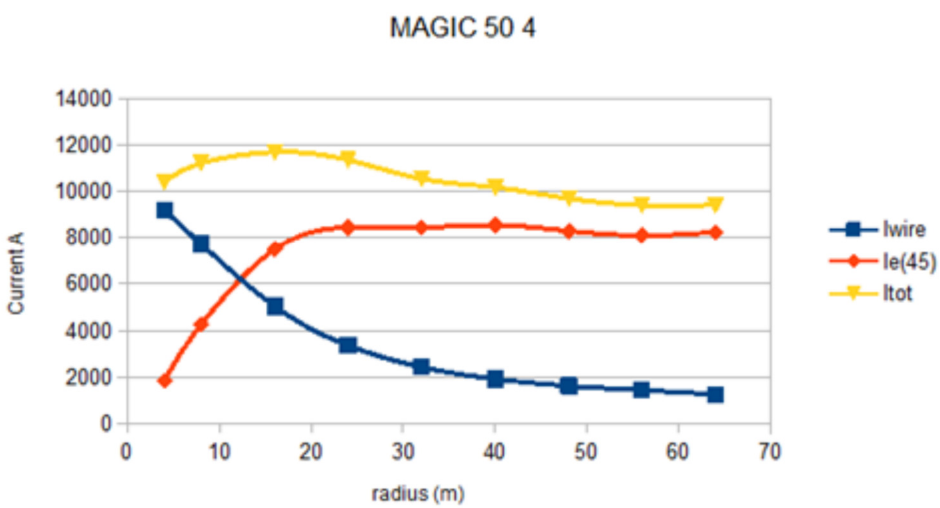
First we will look at time difference of the peak current at the set of diagnostic ranges (4 to 64 m) as a measure of phase difference of the current in the wire over range. Of course, the absolute phase is arbitrary. The phase is nearly linear with range and it covers about 2/3 of a cycle, which is sensible since the wavelength is 100 meters. Again we note that the phase of the wire current is not constant along the wire.



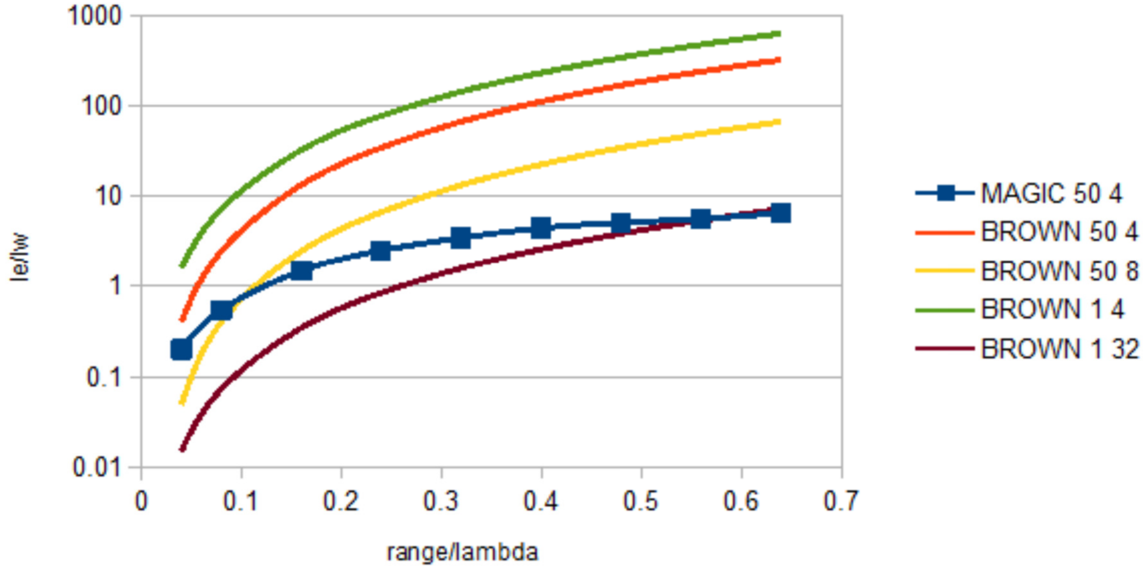
Next we look at the phase difference between the current in the ground (at 45 degrees) and the wire current. Now we plot the phase difference against range. From Brown you might have guessed that this curve would be flat at 90 degrees with the ground current leading the wire current (recall the j). This is not the case, although the ground current does lead the wire current. The earth current at 22.5 degrees shows essentially the same phase behavior as the 45 degree azimuth current. However, as expected, the 22.5 azimuth current tends to turn toward the radial at smaller radii as seen before. There is some jitter in the phase differences likely due to inability to make the radii identical.



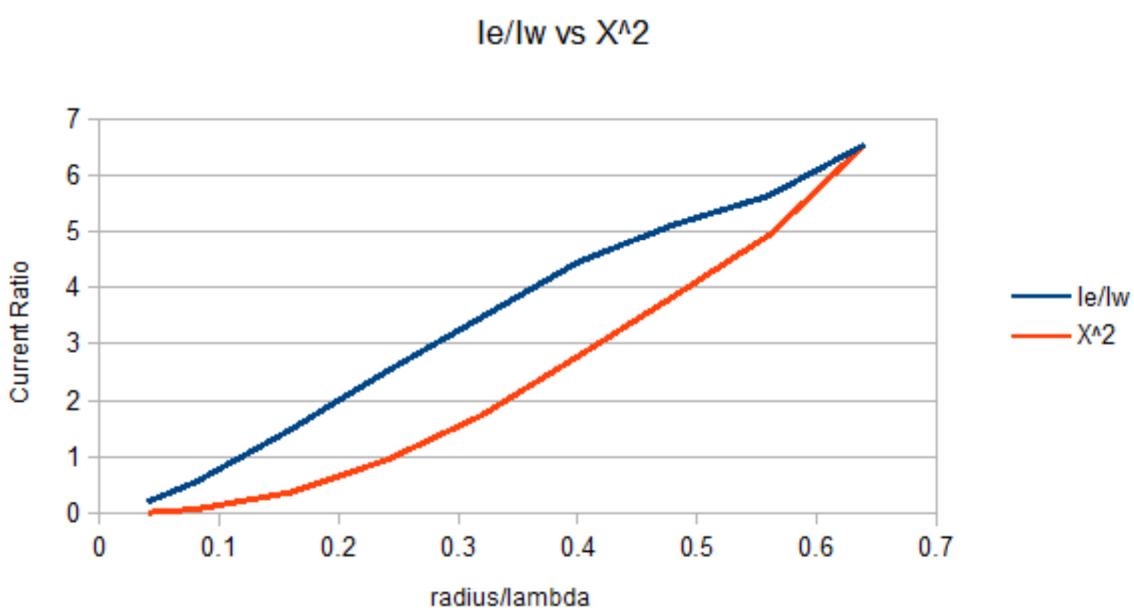
In computing the total current we now take into account the observed phase differences above (rather than 90 degrees). Effects of the phase differences are modest for this. The wire and earth currents for the MAGIC calculation are shown below, where the total current is the sum, phased according to the last plot. In the simulation the normalization of the total current in the antenna is 10.4 kA - this value is arbitrary.



A comparison of the calculation with the Brown model for several selections of radial wire radii (cm) and number of radials (n) are provided to show examples of the ratio of the earth current to the wire current on a log scale. Recall MAGIC used only a 50 cm radius and 4 radials.



It is clear that the MAGIC dependence on range does not match well any of the choices for model parameters and certainly not for the 50,4 case used in MAGIC. The principal difference is in the slope at larger ranges. To illustrate this further, recall that the Brown model for Ie/Iw goes very much like (range)2. The next figure shows that MAGIC result versus X2 when the curves are matched at the largest radius used but a linear plot scale is used. The MAGIC calculation suggests Ie/Iw is nearly linear with range as one might have speculated based on the fraction of the range covered by the radial wire. Still this is a complicated E&M problem so intuition and/or simple arguments may be of very limited value.



Summary of Important Points

The key Brown expression Eq (8) for the ratio of earth current to wire current is of unknown origin with no references.

The only non-funny Brown measurement of Iw for n=15 is not very consistent with Eq (8). [However, the data are from finite length radials while the theory is not – but if the theory is not good for non-infinite radials, why are we doing this?]

Calculation with a high-tone Maxwell equation solver does not appear to agree well with Eq (8) in amplitude or phase. [However, the application of this kind of solver to antenna problems with elements very narrow compared to a wavelength are subject to very legitimate skepticism.]

Some caution should probably be employed in the application of the Brown theory.

Acknowledgment

I thank R. Fry and H. Pfizenmayer for helpful information.

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